

# New branch of Kaluza-Klein compactification

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We found a new branch of solutions in Freund–Rubin type flux compactifications. The geometry of these solutions is described as the external space which has a de Sitter symmetry and the internal space which is topologically spherical. However, it is not a simple form of  $dS_p \times S^q$  but a warped product of de Sitter space and a deformed sphere. We explicitly constructed numerical solutions for a specific case with  $p = 4$  and  $q = 4$ . We show that the new branch of solutions emanates from the marginally stable solution in the branch of  $dS_4 \times S^4$  solutions.

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## I. INTRODUCTION

It is widely believed that de Sitter spacetime well describes our accelerating Universe in the early and also in the present epoch. An intriguing challenge is to realize the Universe as we know it in higher-dimensional theory such as string theory. Therefore, it is interesting to obtain successful embeddings of 4-dimensional de Sitter space in higher-dimensional spacetimes. In order to do so, it is necessary to stabilize the compact extra dimensions. Without stabilization of the extra dimensions, it is impossible to obtain effective 4-dimensional theory in higher dimensional spacetimes.

Freund–Rubin compactification is simple model with a stabilization mechanism by flux [1]. In this model, considering  $(p + q)$ -dimensional spacetime, there is a  $q$ -form flux field for stabilizing the  $q$ -dimensional compact space. Moreover, introducing a bulk cosmological constant allows an external de Sitter space and an internal manifold with positive curvature. We obtain a  $(p + q)$ -dimensional product spacetime of  $p$ -dimensional de Sitter space  $dS_p$  and  $q$ -dimensional sphere  $S^q$ .

If the Hubble parameter is higher than the critical value determined by the flux that stabilizes the extra dimensions, in the scalar sector, massless mode appears at the critical value and this mode eventually becomes tachyonic [2, 3] (also see, for example, [4, 5, 6]). This unstable mode is the homogeneous mode, in other words, the mode corresponding to the change of the radius of the extra dimensions. Physically, this instability can be interpreted based on thermodynamic arguments as follows. For a given total flux integrated over the compact space, which is the conserved quantity of the system, there are some spacetimes with different configurations. We expect that if one configuration is unstable for a fixed total flux, another more stable configurations will exist for the same parameter. Indeed, we define the entropy of the system as one quarter of the total area of de Sitter horizon, and we can show unstable solutions where tachyonic modes appear belonging to an entropically lower branch in a family of solutions. Moreover, the critical value at which the upper branch and lower branch merge is nothing but the one at which a massless mode appears and the spacetime becomes marginally stable. Thus the onset of the thermodynamic instability exactly coincides with that of the dynamical instability [6].

However, it is known that other tachyonic modes emerge as the number of the extra dimensions is larger [2, 3]. These additional tachyonic modes correspond to inhomogeneous Kaluza–Klein excitations with quadrupole and higher multi-pole modes ( $l \geq 2$ ), which indicate deformations of the compact internal space. In order to shed light on its nature and nonperturbative properties, the close connection between thermodynamic and dynamical stability could be expected to play an important role. This new instability suggests the existence of a new branch of solutions at the point where a marginal massless mode appears, as well as for black strings the nonuniform string branch emanates from the uniform string one at the Gregory–Laflamme (GL) point [7, 8]. (See e.g. [9, 10] and references therein.)

The aim of this paper is to examine a new and nontrivial branch of solutions in Freund–Rubin type flux compactifications, which have a de Sitter space and the *warped* compact space with sphere topology rather than a round sphere. In Sec. II we review general Freund–Rubin flux compactifications and show trivial solutions with configuration  $dS_p \times S^q$ . In Sec. III we explore nontrivial warped solutions and show such a numerical solution for  $p = 4$  and  $q = 4$  explicitly.

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## II. FREUND–RUBIN SOLUTION

We briefly review general Freund–Rubin flux compactifications with  $dS_p \times S^q$  and dynamical stability of those configurations. The  $(p+q)$ -dimensional action is

$$I = \frac{1}{16\pi} \int d^{p+q}x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{q!} F_{(q)}^2 \right), \quad (1)$$

where  $F_{(q)}$  is the  $q$ -form field for stabilizing the  $q$ -sphere and  $\Lambda$  is a  $(p+q)$ -dimensional cosmological constant. The equations of motion are

$$G_{MN} = \frac{1}{(q-1)!} F_{ML_1 \dots L_{q-1}} F_N{}^{L_1 \dots L_{q-1}} - \frac{1}{2q!} F_{(q)}^2 g_{MN} - \Lambda g_{MN}, \quad (2)$$

and

$$\nabla_M F^{MN_1 \dots N_{q-1}} = 0. \quad (3)$$

For the trivial solution of Eqs. (2) and (3), the metric and the  $q$ -form flux are given by

$$ds^2 = -dt^2 + e^{2ht} d\vec{x}_{p-1}^2 + \rho^2 d\Omega_q^2, \quad (4)$$

and

$$F_{(q)} = b \epsilon_{\mu_1 \dots \mu_q}, \quad (5)$$

where  $\epsilon_{\mu_1 \dots \mu_q}$  is the volume element of the  $q$ -sphere with a radius  $\rho$ . Also  $h$  and  $b$  are the Hubble parameter of  $p$ -dimensional de Sitter space and the flux strength, respectively. From (4) and (5), the Einstein and Maxwell equations reduce to two algebraic equations:

$$\begin{aligned} (q-1)\rho^{-2} - (p-1)h^2 &= b^2, \\ (q-1)^2\rho^{-2} + (p-1)^2h^2 &= 2\Lambda. \end{aligned} \quad (6)$$

From the above equations, we obtain a relation in terms of two parameters  $b$  and  $h$  when  $dS_p \times S^q$  solutions exist:

$$(p-1)(p+q-2)h^2 + (q-1)b^2 = 2\Lambda. \quad (7)$$

The analysis of linear perturbations in this background spacetime show that there are two channels of instabilities in the scalar sector with respect to the  $p$ -dimensional de Sitter symmetry: (i) homogeneous excitation and (ii) inhomogeneous excitation with quadrupole and higher multi-pole moments of Kaluza–Klein modes [2, 3]. (i)  $l = 0$  mode is a so-called volume modulus and becomes unstable as the Hubble parameter  $h$  of the de Sitter space becomes very large (i.e., the flux  $b$  small). This seems to be a generic feature of de Sitter compactifications [4, 5]. The stability condition that mass squared of the  $l = 0$  mode will not be tachyonic is given by

$$h^2 \leq \frac{2\Lambda(p-2)}{(p-1)^2(p+q-2)} \quad \text{or} \quad b^2 \geq \frac{2\Lambda}{(p-1)(q-1)}. \quad (8)$$

Furthermore, as mentioned in the introduction, we can derive this threshold according to the thermodynamic argument using entropy [6].

On the other hand, instabilities arising from (ii)  $l \geq 2$  modes appear when the number of extra dimensions is higher, to be precise,  $q \geq 4$ . The stability condition that the  $l = 2$  tachyonic mode should not appear is

$$h^2 \geq \frac{2\Lambda[2+q-3pq+(p-1)q^2]}{q(q-3)(p-1)^2(p+q-2)} \quad \text{or} \quad b^2 \leq \frac{4\Lambda}{q(q-3)(p-1)}. \quad (9)$$

Finally, let us summarize the stability of Freund–Rubin solutions with  $dS_p \times S^q$ . For  $q = 2$  and  $q = 3$  there is only one channel of instability arising from  $l = 0$  mode. The solutions for fluxes higher than the critical value given by (8) are stable. For  $q = 4$ , an additional tachyonic mode for  $l = 2$  appears and stable configurations are allowed in the range of (8) and (9). For  $q \geq 5$ , there is no stable  $dS_p \times S^q$  solution for any value of fluxes.

### III. NEW BRANCH

In this section we explore a warped solution such that the  $p$ -dimensional external space have a de Sitter symmetry with a warp factor depending on internal coordinates and the  $q$ -dimensional internal space is a deformed sphere. The  $(p+q)$ -dimensional geometry of such solutions is generally described as the metric form

$$ds^2 = A(y)^2[-dt^2 + e^{2ht}d\vec{x}_{p-1}^2] + g_{ab}(y)dy^a dy^b, \quad (10)$$

where  $A(y)^2$  is the warp factor depending on internal coordinates  $y^a$ , and  $g_{ab}$  is the  $q$ -dimensional metric of the internal compact manifold which is topologically  $S^q$ . For simplicity, we assume that the internal space has  $SO(q)$  isometry, then the  $(p+q)$ -dimensional metric reduces to

$$ds^2 = A(r)^2[-dt^2 + e^{2ht}d\vec{x}_{p-1}^2] + B(r)^2dr^2 + C(r)^2d\Omega_{q-1}^2. \quad (11)$$

As there is still coordinate freedom in this metric, we can choose

$$A(r) = e^{\Phi(r)}, \quad B(r) = e^{-\frac{p}{q-2}\Phi(r)}, \quad C(r) = e^{-\frac{p}{q-2}\Phi(r)}a(r). \quad (12)$$

Note that this choice makes the problem similar to that of Friedmann–Robertson–Walker cosmology with a scalar field (see appendix and for an example, [11]). Consequently, the metric form is given by

$$ds^2 = e^{2\Phi(r)}[-dt^2 + e^{2ht}d\vec{x}_{p-1}^2] + e^{-\frac{2p}{q-2}\Phi(r)}[dr^2 + a^2(r)d\Omega_{q-1}^2], \quad (13)$$

and the  $q$ -form flux is

$$F_{(q)} = be^{-\frac{2p(q-1)}{q-2}\Phi}a^{q-1}dr \wedge d\Omega_{q-1}, \quad (14)$$

where  $b$  is a constant. This  $q$ -form field satisfies the Maxwell equation (3) and the Bianchi identity  $dF_{(q)} = 0$ . It is worth noting that the values of two parameters  $b$  and  $h$  are not always globally meaningful quantities, because physical quantities locally observed depend on the warp factor. Here we have taken the forms in order that meanings of  $b$  and  $h$  should become the same as in the previous definitions in Eqs. (4) and (5) which is realized under a trivial solution  $\Phi = 0$ .

Now we consider  $p = 4$  and  $q = 4$  case for an explicit example. This is because Freund–Rubin solutions for  $q \geq 4$  have the instability arising from higher multi-pole modes, as we have seen. In this case the equations of motion are given by

$$\begin{aligned} \frac{a''}{a} &= -6\Phi'^2 + \frac{a'^2 - 1}{a^2}, \\ \Phi'' &= 3h^2e^{-6\Phi} + \frac{1}{2}b^2e^{-12\Phi} - \frac{1}{3}e^{-4\Phi} - 3\frac{a'}{a}\Phi' \\ &= -3\Phi'^2 + \frac{1}{4}b^2e^{-12\Phi} + \frac{1}{6}e^{-4\Phi} - 3\frac{a'}{a}\Phi' + \frac{3}{2}\frac{a'^2 - 1}{a^2}, \end{aligned} \quad (15)$$

where the prime denotes the derivative with respect to  $r$ . Note that we rescaled  $b \rightarrow b\Lambda^{1/2}$ ,  $h \rightarrow h\Lambda^{1/2}$ ,  $a \rightarrow a\Lambda^{-1/2}$  and  $r \rightarrow r\Lambda^{-1/2}$  to normalize  $\Lambda$  to unity. The constraint equation is given by

$$3 \left[ \left( \frac{a'}{a} \right)^2 - \frac{1}{a^2} \right] = 6\Phi'^2 + 6h^2e^{-6\Phi} + \frac{b^2}{2}e^{-12\Phi} - e^{-4\Phi}. \quad (16)$$

In order to solve these equations we will require boundary conditions. We assume the internal space is topologically spherical and  $a = 0$  represents poles of the sphere. Hence, at a pole ( $a = 0$ ) boundary conditions are imposed to ensure that it is regular, and at equatorial plane other boundary conditions are imposed for the internal space to be symmetric with respect to this plane. We note that it is an assumption for the internal space to be symmetric. This is because analyses of linear perturbations imply that an inhomogeneous unstable mode first appears at a quadrupole ( $l = 2$ ) mode, so that we expect to find a corresponding solution to the  $l = 2$  mode in linear perturbations.

We suppose that the equatorial plane is located at  $r = 0$ , then boundary conditions at  $r = 0$  are given by  $\Phi(0) = 0$ ,  $\Phi'(0) = 0$  and  $a'(0) = 0$ . Note that although a value of  $\Phi(0)$  is arbitrary, we have set  $\Phi(0) = 0$  for simplicity here. Moreover, from the constraint equation (16) we obtain a value of  $a$  at  $r = 0$ :

$$a(0) = \sqrt{\frac{6}{2 - b^2 - 12h^2}}. \quad (17)$$

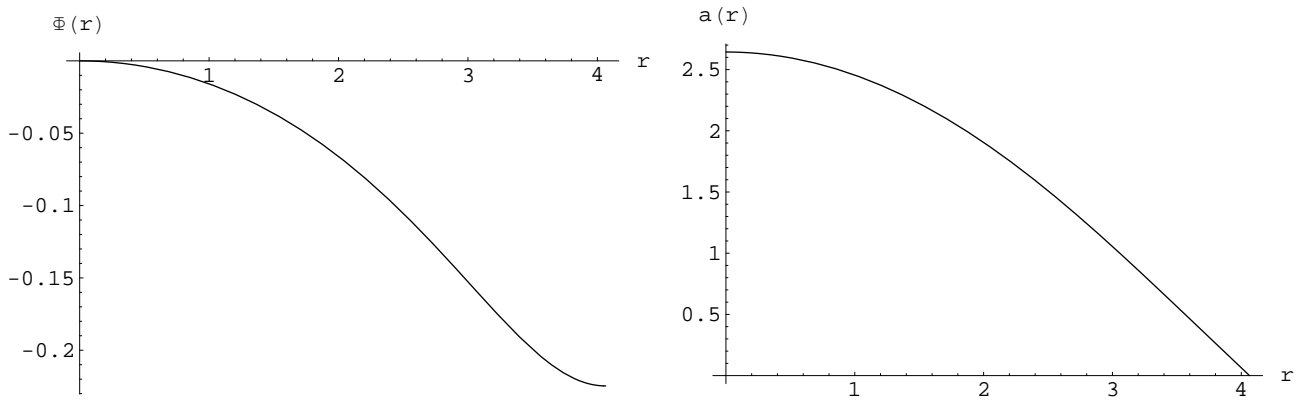


FIG. 1: The numerical solutions of  $\Phi$  (left) and  $a$  (right) when  $b = 1/\sqrt{15}$  and  $h = 0.29928$ .  $r = 0$  represents the equatorial plane and the pole is located at  $r$  where  $a(r) = 0$ .

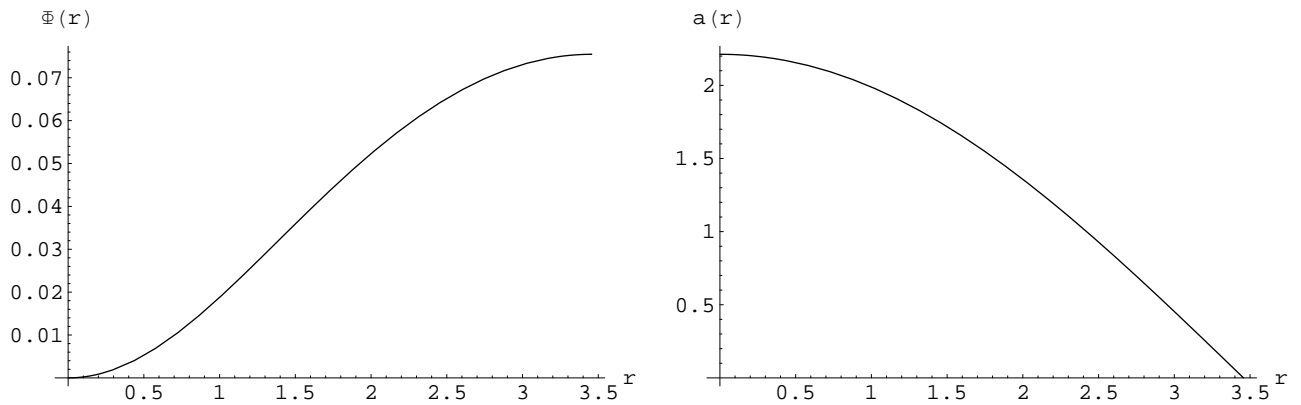


FIG. 2: The numerical solutions of  $\Phi$  (left) and  $a$  (right) when  $b = \sqrt{11/15}$  and  $h = 0.058325$ .

Changing values of  $b$  and  $h$  as shooting parameters, we numerically integrate the ODEs of (15) from  $r = 0$  to  $r > 0$  to obtain the desired solutions satisfying the boundary conditions using the shooting method. As we have mentioned, since we are looking for solutions which have regular compact internal space, at the other side we require boundary conditions  $a = 0$ ,  $a' = 1$  and  $\Phi' = 0$  from regularity at the pole.

As a result, we have two families of solutions. One is a family of well-known, trivial solutions given by

$$\Phi(r) = 0, \quad a(r) = \rho \cos \frac{r}{\rho} \quad (18)$$

where  $\rho = a(0)$  from (17) with satisfying the relation  $18h^2 + 3b^2 = 2$  to which Eq. (7) leads. The other is a family of nontrivial solutions. For example, numerical solutions for some values of  $b$  and  $h$  are shown in Fig. 1 and 2. Clearly  $\Phi(r)$  is not constant, so that this solution is a warped one. Figure 3 shows two families of solutions in terms of two parameters  $b$  and  $h$ . We see from this figure that the branch of trivial solutions and that of nontrivial, warped solutions intersect at one point  $(b^2, h^2) = (\frac{1}{3}, \frac{1}{18})$ , where there is only a trivial solution. What is especially important is that a massless mode for  $l = 2$  appears on this point which is given by (9) according to linear perturbations. This means that the new branch of warped solutions emanates from the marginally stable unwarped solution as we have expected.

#### IV. CONCLUSION

In this paper we have studied new solutions in Freund–Rubin type flux compactifications. By numerical analysis we found warped solutions with a deformed sphere of the internal space. A new branch of these solutions intersects

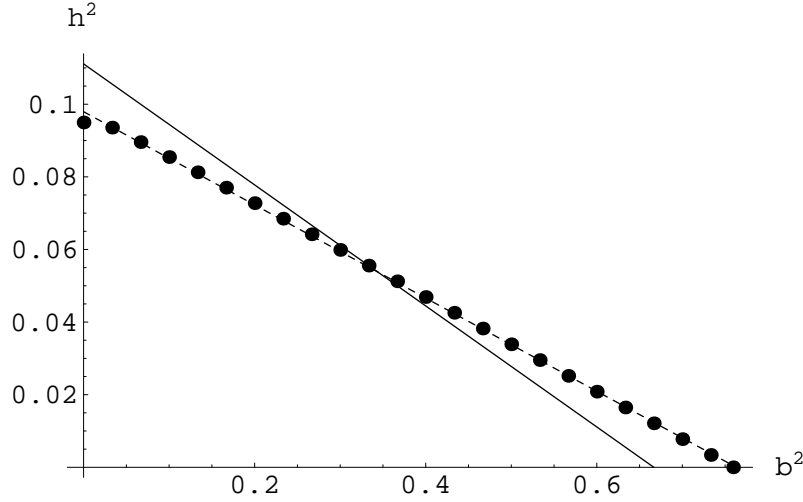


FIG. 3: Two branches of solutions in the  $(b^2, h^2)$  space. The solid line represents the branch of Freund–Rubin solutions. The bold points correspond to values calculated numerically for which warped solutions have been found. It is likely that the branch of warped solutions would have a linear relation (dashed line) between  $b^2$  and  $h^2$  also.

the branch of Freund–Rubin solutions at a point in  $(b^2, h^2)$ -space. The parameters represented by this point agree with those of the spacetime configuration that massless mode for  $l = 2$  in the scalar sector emerges under linear perturbations. This is very similar to the case of black strings in which the uniform string branch and the nonuniform branch meet at the GL point where the GL instability occurs. Therefore, it seems quite probable that for higher multi-pole mode instability there is the close connection between thermodynamic and dynamical stability. We hope that there are rich phase (thermodynamic) structures not only for black objects, but for de Sitter compactifications. Of course, we do not know the stability of new branch of solutions yet. However, the existence of another branch of solutions implies that unstable configurations for large values of fluxes may be stabilized by deformation of the internal geometry and change of flux distribution on the sphere. Also, even if the number of the extra dimensions is too large for stable  $dS_p \times S^q$  solutions to exist, there may be stable configurations in the warped branch. We will reveal thermodynamic properties of this system in a forthcoming work. It would provide evidence that a wider class of gravitating systems other than black objects can be applied to the *correlated stability conjecture* [12, 13].

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### APPENDIX A

The equations of motion are

$$\begin{aligned} \frac{a''}{a} &= \frac{b^2}{q-2} e^{-\frac{2p(q-1)}{q-2}\Phi} + \frac{p(p-1)}{q-2} h^2 e^{-\frac{2(p+q-2)}{q-2}\Phi} - \frac{2\Lambda}{q-2} e^{-\frac{2p}{q-2}\Phi} - (q-2) \frac{a'^2 - 1}{a^2} \\ &= -\frac{p(p+q-2)}{(q-2)^2} \Phi'^2 + \frac{a'^2 - 1}{a^2}, \\ \Phi'' &= (p-1) h^2 e^{-\frac{2(p+q-2)}{q-2}\Phi} + \frac{q-1}{p+q-2} b^2 e^{-\frac{2p(q-1)}{q-2}\Phi} - \frac{2\Lambda}{p+q-2} e^{-\frac{2p}{q-2}\Phi} - (q-1) \frac{a'}{a} \Phi'. \end{aligned} \quad (\text{A1})$$

The constraint equation is given by

$$\frac{(q-1)(q-2)}{2} \left[ \left( \frac{a'}{a} \right)^2 - \frac{1}{a^2} \right] = \frac{p(p+q-2)}{2(q-2)} \Phi'^2 + \frac{p(p-1)}{2} h^2 e^{-\frac{2(p+q-2)}{q-2}\Phi} + \frac{b^2}{2} e^{-\frac{2p(q-1)}{q-2}\Phi} - \Lambda e^{-\frac{2p}{q-2}\Phi}. \quad (\text{A2})$$

This system of equations can be rewritten as follows:

$$\begin{aligned}\frac{d\Phi}{dr} &= \Psi, \\ \frac{d\Psi}{dr} &= -(q-1)\mathcal{H}\Psi - \frac{q-2}{p(p+q-2)}V_{,\Phi}(\Phi), \\ \frac{d\mathcal{H}}{dr} &= -\mathcal{H}^2 - \frac{p(p+q-2)}{(q-1)(q-2)}\Psi^2 + \frac{2}{(q-1)(q-2)}V(\Phi),\end{aligned}\tag{A3}$$

where we define “Hubble parameter” as

$$\mathcal{H} \equiv \frac{a'}{a},\tag{A4}$$

and “potential” as

$$V(\Phi) \equiv \frac{p(p-1)}{2}h^2e^{-\frac{2(p+q-2)}{q-2}\Phi} + \frac{b^2}{2}e^{-\frac{2p(q-1)}{q-2}\Phi} - \Lambda e^{-\frac{2p}{q-2}\Phi}.\tag{A5}$$

Also, the constraint equation becomes

$$\frac{(q-1)(q-2)}{2}\mathcal{H}^2 - \frac{p(p+q-2)}{2(q-2)}\Psi^2 - V(\Phi) = \frac{(q-1)(q-2)}{2}\frac{1}{a^2},\tag{A6}$$

which we can regard as “Friedmann equation” with a scalar field  $\Phi$ .

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